Summary of Chapter 3:

(1) Cantor Set: Construction, length, cardinality, dimension.

(2) ε-neighborhood (abbr. ε-nbhd):

$$V_\varepsilon(a) = \{x : |x-a| < \varepsilon\}$$

(3) open set: $\forall a \in O$, $\exists$ ε-nbhd $V_\varepsilon(a) \subseteq O$

Thm: The union of arbitrarily many open sets are open.

The intersection of finitely many open sets is open.

(4) Limit Point: $\forall \varepsilon > 0$, $\exists\, a \in A$, $a \neq x$, $a \in V_\varepsilon(x)$

Equivalently: $\forall \varepsilon > 0$, $\exists$ infinitely many $a \in A$, $a \in V_\varepsilon(x)$

Equivalently: $\exists (a_n)$ contained in A, $x = \lim_{n\to\infty} a_n$.

Isolated Point: $x \in A$ not a limit point

Equivalently: $\exists\, \varepsilon > 0$, $V_\varepsilon(x) \cap A = \{x\}$.

(5) Closed Set: x is a limit point of $F \Rightarrow x \in F$

Thm: $F \subseteq \mathbb{R}$ is closed $\Leftrightarrow$ $\forall (a_n)$ Cauchy seq. in F, $\lim_{n\to\infty} a_n \in F$

Closure of a subset: $\bar{A} = A \cup \{\text{limit points of } A\}$

Thm: $\bar{A}$ = smallest closed set containing A.

(6) Complement: $A^c = \{x \notin A\}$

Thm: O open $\Leftrightarrow$ O^c closed

F closed $\Leftrightarrow$ F^c open.

Thm The intersection of arbitrarily many closed sets is closed.

The union of finitely many closed sets is closed.

(7) Compact Set: $\forall (a_n)$ in K, $\exists (a_{n_k})$ subsequence, $\lim_{k\to\infty} a_{n_k} \in K$.

Bounded Set: $\exists M>0$, $\forall a \in A$, $|a| \le M$.

Thm: K is compact $\Leftrightarrow$ K is closed and bounded

Thm: Nested Compact Set Property:

$K_1 \supseteq K_2 \supseteq \cdots$ nested sequence of compact sets,

$\Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \phi$

(8) Open cover: $\{O_\lambda \text{ open} : \lambda \in \Lambda\}$ satisfying $\bigcup_{\lambda\in\Lambda} O_\lambda \supseteq A$

Finite subcover: $\{O_{\lambda_1}, O_{\lambda_2}, \cdots, O_{\lambda_k} : \lambda_1, \cdots, \lambda_k \in \Lambda\}$, $\bigcup_{i=1}^{k} O_{\lambda_i} \supseteq A$.

Heine-Borel Thm: TFAE

(i) K is compact.

(ii) K is closed and bounded.

(iii) Every open cover for K has a finite subcover.

(iv) The contrapositive of (iii). (Formulate yourself).

(9) Perfect Set: closed set w/o isolated points

Thm: P nonempty, perfect $\Rightarrow$ P is uncountable.

(10) Separated Sets: $\bar{A} \cap B = A \cap \bar{B} = \phi$

Disconnected Set: $E = A \cup B$, A, B nonempty, separated.

Connected Set: not disconnected.

Thm: E is connected $\Leftrightarrow$ $\forall A, B \neq \phi$ s.t. $E = A \cup B$

$\exists (x_n)$ convergent sequence in A (or B)

$(x_n) \to x$, $x \in B$ (or A)

Thm: $E \subseteq \mathbb{R}$ is connected $\Leftrightarrow$ $\forall a, b \in E$, s.t. $a<b$, $[a,b] \subseteq E$.

(11) F_σ-set : countable union of closed sets

G_δ-set : countable intersection of open sets.

Dense Set : $\forall a < b$, $\exists x \in G$, $a < x < b$.

Equivalently : $\forall O$ open set, $G \cap O \neq \phi$.

Equivalently : $\overline{G} = \mathbb{R}$.

Thm : $\{G_n \subseteq \mathbb{R}$ dense open : $n = 1, 2, \cdots\}$

$$\Rightarrow \bigcap_{n=1}^{\infty} G_n \neq \phi.$$

Nowhere-Dense Set : $\overline{E}$ contains no nonempty open intervals.

Baire's Theorem : $\mathbb{R}$ cannot be written as a countable union of nowhere-dense sets

HW to be graded: 3.2.4. 3.2.6.* 3.2.13. 3.3.3.* 3.3.5.

grade will be given for free.

My sol'n to 3.2.13

Suppose $A \neq \phi$ is both open and closed. We prove that $A = \mathbb{R}$.

$A \neq \phi \Rightarrow \exists\, a \in A$

A open $\Rightarrow \exists\, \varepsilon > 0$, $(a-\varepsilon, a+\varepsilon) \subseteq A$.

A closed $\Rightarrow a+\varepsilon \in A$.

We are going to exploit the above argument.

Set $x = \sup \{ b : (a-\varepsilon, b) \subseteq A \}$

If x is finite, then $\forall n > 0$, $\exists\, b_n > x - \frac{1}{n}$, $(a-\varepsilon, b_n) \subseteq A$.

A is closed $\Rightarrow b_n \in A$

$x - \frac{1}{n} < b_n \leq x \Rightarrow (b_n) \to x$

A is closed, each $b_n \in A \Rightarrow x \in A$.

A is open $\Rightarrow \exists\, \varepsilon' > 0$, $(x-\varepsilon', x+\varepsilon') \subseteq A$

$\Rightarrow (a-\varepsilon, x+\varepsilon') \subseteq A$

Therefore, we proved that if $x = \sup \{ b : (a-\varepsilon, b) \subseteq A \}$ is finite then $x+\varepsilon'$ makes $(a-\varepsilon, x+\varepsilon') \subseteq A$, contradicting the choice of x.

So $x = +\infty$. i.e. $(a-\varepsilon, +\infty) \subseteq A$.

Similarly, if $y = \inf \{ c : (c, +\infty) \subseteq A \}$ then $y = -\infty$.

Therefore $\mathbb{R} \subseteq A$.

Remark: You'll see a similar argument when you study ODEs and flows in any proof-based class containing this topic.

Another solution:

Let $\phi \neq A \subsetneqq \mathbb{R}$ be a proper subset that is both open and closed.

Then $\phi \neq A^c \subsetneqq \mathbb{R}$ is also a both open and closed set.

Then $\mathbb{R} = A \cup A^c$.

Now A is closed $\Rightarrow \bar{A} = A \Rightarrow \bar{A} \cap A^c = A \cap A^c = \phi$

A^c is closed $\Rightarrow \overline{A^c} = A \Rightarrow A \cap \overline{A^c} = A \cap A^c = \phi$.

Therefore A, A^c form a separation of $\mathbb{R} \Rightarrow \mathbb{R}$ disconnected.

But $\mathbb{R}$ is connected by Theorem 3.4.7 (applied to $E = \mathbb{R}$).

Contradiction. So if A is a proper subset, A cannot be both open & closed

$\Rightarrow$ The only both open and closed subset of $\mathbb{R}$ are $\mathbb{R}$, ϕ.

Remark: This is a standard topological proof.

My solution to 3.2.6.

(a) $(-\infty, \sqrt{2}) \cup (\sqrt{2}, +\infty)$ is open and contains $\mathbb{Q}$.

(b) Consider $[1, \infty) \supseteq [2, \infty) \supseteq \cdots \supseteq [n, \infty) \supseteq \cdots$

The intersection of these closed sets is empty.

(c) True by density thm: $A \neq \phi \Rightarrow \exists a \in A$

A open $\Rightarrow \exists \varepsilon > 0$, $(a-\varepsilon, a+\varepsilon) \subseteq A$.

Density theorem $\Rightarrow \exists r \in \mathbb{Q}$, $r \in (a-\varepsilon, r+\varepsilon) \Rightarrow r \in A$

(d) Consider $\{\sqrt{2}\} \cup \{\sqrt{2} + \frac{1}{n}\}$. The limit point of the set is $\sqrt{2}$, since any subsequence in the set converges to $\sqrt{2}$. So the set is closed. Easy to see it's bounded and contains no rational numbers.

(e) (not graded). Cantor set is closed, since each C_n is closed and the intersection of arbitrary closed sets is closed.

My solution to 3.3.3. $K \subseteq \mathbb{R}$ is closed and bounded $\Rightarrow K$ is compact.

Pf: Take an arbitrary sequence (a_n) in K.

K is bounded $\Rightarrow$ (a_n) is bounded

Bolzano-Weierstrass Thm $\Rightarrow$ $\exists\, (a_{n_k})$ subsequence of (a_n), $\exists\, a \in \mathbb{R}$, $a_{n_k} \to a$

In particular, a is a limit point of K.

So by K being closed, $a \in K$.

Therefore we proved that

$\forall$ seq. in (a_n) contains a subseq. (a_{n_k}) that converges to a limit in K.

My solution to 3.2.10c. Why there cannot exist a set with an uncountable number of isolated points?

(1) Note that $\mathbb{R}$ can be expressed as a countable union of closed intervals

i.e. $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} I_n$, $I_n = [n, n+1]$

(2) If A is such a set, then $\exists\, n \in \mathbb{Z}$, $I_n \cap A$ is also such a set.

Otherwise, $I_n \cap A$ has at most countably many isolated points, $\forall n \in \mathbb{Z}$

$\Rightarrow$ Isolated points in $A = \bigcup_{n \in \mathbb{Z}} I_n \cap A$ are at most countable b/c a countable union of countable sets is countable.

So WLOG, we can discuss the problem within an interval I_n.

(3) Lemma: If $\{a_i : i\in I\}$ is an uncountable set of nonnegative numbers with $\sum_{i\in I} a_i < \infty$, then there is at most countable many a_i's are nonzero, i.e. $a_i = 0$ for all but countably many $i \in I$.

Pf: Let $\sum_{i\in I} a_i = M < \infty$. Consider the set $S_n = \{i\in I : a_i > \frac{1}{n}\}$, $n \in \mathbb{Z}_+$

Then $M \geq \sum_{i\in S_n} a_i > \frac{1}{n} \cdot \# S_n \Rightarrow \# S_n \leq nM$

$\Rightarrow S_n$ is a finite set.

$\Rightarrow \bigcup_{n=1}^{\infty} S_n$ is at most countable.

Note that $\{i\in I : a_i > 0\} = \bigcup_{n=1}^{\infty} S_n$

($i \in LHS \Leftrightarrow a_i > 0 \Leftrightarrow \exists n,\ a_i > \frac{1}{n}$ (Archimedean Property)

$\Leftrightarrow \exists n,\ i\in S_n \Rightarrow i \in RHS$)

(4) $I_n \cap A$ cannot have uncountably many isolated points. $\forall n \in \mathbb{Z}$.

Let $X = \{x_\lambda : \lambda \in \Lambda\}$ be the set of isolated points in $I_n \cap A$.

WLOG, $n \notin X$, $n+1 \notin X$.

Then $\forall \lambda \in \Lambda$, $\exists \varepsilon_\lambda' > 0$, $(x_\lambda - \varepsilon_\lambda', x_\lambda + \varepsilon_\lambda') \cap X = \{x_\lambda\}$.

By shrinking ε_λ' one can confine $(x_\lambda - \varepsilon_\lambda', x_\lambda + \varepsilon_\lambda')$ in I_n.

(This is always possible if $x_\lambda \neq n$, $x_\lambda \neq n+1$)

Moreover, $\forall \mu \in \Lambda$, $x_\mu \neq x_\lambda$

$\Rightarrow x_\mu \notin (x_\lambda - \varepsilon_\lambda', x_\lambda + \varepsilon_\lambda'),\ x_\lambda \notin (x_\mu - \varepsilon_\mu', x_\mu + \varepsilon_\mu')$

$\Rightarrow |x_\mu - x_\lambda| > \varepsilon_\lambda'.\ |x_\mu - x_\lambda| > \varepsilon_\mu'$

$\Rightarrow |x_\mu - x_\lambda| > \frac{1}{2}(\varepsilon_\lambda' + \varepsilon_\mu')$

So if we pick $\varepsilon_\lambda = \frac{1}{2}\varepsilon_\lambda'$, then all $(x_\lambda - \varepsilon_\lambda, x_\lambda + \varepsilon_\lambda)$'s are disjoint

Therefore the length of $\bigcup_{\lambda\in\Lambda} (x_\lambda - \varepsilon_\lambda, x_\lambda + \varepsilon_\lambda)$ is $\sum_{\lambda\in\Lambda} 2\varepsilon_\lambda$.

Since $\forall \lambda \in \Lambda, (x_\lambda - \varepsilon_\lambda, x_\lambda + \varepsilon_\lambda) \subseteq I_n \Rightarrow \bigcup_{\lambda \in \Lambda} (x_\lambda - \varepsilon_\lambda, x_\lambda + \varepsilon_\lambda) \subseteq I_n$

So the length of $\bigcup_{\lambda \in \Lambda} (x_\lambda - \varepsilon_\lambda, x_\lambda + \varepsilon_\lambda)$ should be no larger than 1.

By lemma in (3), there are at most countably many λ s.t. $\varepsilon_\lambda > 0$.

$\Rightarrow$ at most countably many isolated points.